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A non-local hidden-variable model that violates Leggett-type inequalities

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Abstract

Recent experiments of Gröblacher *et al* proved the violation of a Leggett-type inequality that was claimed to be valid for a broad class of non-local hidden-variable theories. The impossibility of constructing a non-local and realistic theory, unless it entails highly counterintuitive features, seems thus to have been experimentally proved. This would bring us close to a definite refutation of realism. Indeed, realism was proved to be also incompatible with locality, according to a series of experiments testing Bell inequalities. The present paper addresses the said experiments of Gröblacher *et al* and presents an explicit, contextual and realistic, model that reproduces the predictions of quantum mechanics. It thus violates the Leggett-type inequality that was established with the aim of ruling out a supposedly broad class of non-local models. We can thus conclude that plausible contextual, realistic, models are still tenable. This restates the possibility of a future completion of quantum mechanics by a realistic and contextual theory which is not in a class containing only highly counterintuitive models. The class that was ruled out by the experiments of Gröblacher *et al* is thus proved to be a limited one, arbitrarily separating models that physically belong in the same class.

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1. Introduction

Recently, Gröblacher *et al* [1] reported experiments performed with two entangled photons, showing the violation of a Leggett-type inequality that was derived by the said authors. According to them, such a violation rules out a broad class of hidden-variable (HV) models based on non-local realism. This would be an important step toward answering the question about the completeness of quantum mechanics (QM), a question that was raised by Einstein, Podolsky and Rosen in their celebrated paper of 1935. Several experiments testing Bell inequalities served the purpose of closing the detection and the locality loopholes, and for many—though not for all [2]—researchers in the field, 'it is reasonable to consider the violation of local realism as well established fact' [1]. Accordingly, a completion of QM through a HV theory would require that this theory should be based upon the concept of non-locality. Theories of this kind are sometimes taken on an equal basis as 'contextual' theories and, in fact, the Leggett-type inequality that was tested by Gröblacher *et al* does not distinguish contextuality from non-locality. It simply admits as a possibility that measurement outcomes may depend on parameters in distant regions, without specifying whether the involved separations are spacelike or timelike. The class defined by the Leggett-type inequality may include certain non-local realistic theories, but the experiments performed by Gröblacher *et al* were not able to distinguish these theories from local contextual ones. Indeed, for the events involved (photon detection or emission) the time coordinate is not registered and the settings remain fixed. Hence, one cannot tell whether spacetime intervals $\Delta s^2 \equiv c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$ between pairs of events are timelike, spacelike, or light-like, excepting the restricted class of simultaneous events ($\Delta t = 0$, e.g., coincidences in photon detections), for which they are spacelike. Strictly speaking, there is a difference between 'non-locality' and 'contextuality'. 'Non-locality' is often understood in a relativistic sense (causal influences can travel faster than light), while 'contextuality' refers to the fact that a measurement outcome or the state of a system may generally depend on the context in which the measurement takes place or the system is prepared[1]. Now, the ambiguity concerning 'non-local' and 'contextual' models can be removed if we ask the model maker to consider changes in the context and to tell after how much time these changes should manifest themselves in the measurement results. Whatever the answer, the model will be either local or non-local. Note that 'contextual-local' theories are not excluded from the outset, as it might appear at first sight. Indeed, a relativistic theory may admit contextuality if it requires fixing boundary conditions to solve its fundamental equations. Physically, one assumes that these conditions have been fixed well in advance so as to allow causal influences to propagate, without conflicting relativity, from the boundaries toward those parts of the system that are the subject of the fundamental equations. If we instead consider boundary conditions that change with time, then we are faced with the choice between locality and non-locality. Here we will mostly use the term contextuality rather than non-locality, because the cases we shall address do not distinguish whether the spacetime intervals between events (e.g., measurements) are timelike or spacelike. The recently reported experiments [1] are said to exclude a broad class of non-local HV models. Because the experimental arrangement of Gröblacher *et al* could not distinguish local from non-local influences, we prefer to use the term 'contextual' rather than 'non-local' when referring to it. Anyhow, Gröblacher *et al* arrived at the conclusion that any possible completion of QM through a realistic HV theory would be a highly counterintuitive one, entailing exotic features such as a departure from Aristotelian logic, actions into the past, etc. Here we show that the class of contextual models addressed in the experiments of Gröblacher *et al* is—from a physical point of view—a rather limited one. Indeed, as we shall see, this class arbitrarily excludes a type of contextual models that should have been addressed, for the approach to be physically self-consistent. The exclusion of these models invalidates, in fact, the claim of Gröblacher *et al*. To show this, we will present an explicit, contextual model that violates the Leggett-type inequality that the reported experiments put to the test. Of course, the Leggett-type inequality remains true for the restricted class considered by Gröblacher *et al*. Our claim is that this class is physically unsound, for the following reason. Gröblacher *et al* derived a Leggett-type inequality for a class of non-local models in

[1] The term 'non-local' is nevertheless often used in a non-relativistic sense also, for example when referring to quantum correlations, these correlations being not necessarily attributed to superluminal causal influences.

which there are two measuring devices (e.g., polarizers) and a source (of entangled pairs). Furthermore, following Leggett, they relaxed the locality assumption for the polarizers, so that measurement outcomes at one device could depend on the settings of the other. However, the source was supposed to be somehow ‘protected’ from any non-local influences, a condition that appears to be physically unsound once we have relaxed the locality assumption. In the present case, we will assume non-locality for the source as well as for the polarizers. To be sure, Leggett’s original assumption of a source being exclusively influenced by conditions on its neighborhood could be—in principle—experimentally realized, as we will discuss next, but the experiments of Gröblacher *et al* were not designed to fulfil this requirement.

Leggett’s assumption of a ‘protected’ source, referred to above, could be fulfilled in experiments with variable polarizers. Such experiments could test non-local models besides contextual ones. Let us first consider testing contextual local hidden-variable models. Within such a framework we could conceive an arrangement for which the settings of the polarizers constitute events that are spacelike separated from the emission at the source. In this way, the source becomes effectively ‘protected’ from non-local influences and the distribution describing it (see below) can be assumed to be independent of a context that is beyond its immediate neighborhood. Let us consider next non-local models. In this case, though we abandon the framework of special relativity, it is still reasonable to assume that we may define an absolute future with respect to a given event in some reference frame, e.g., the laboratory. Under this assumption, we could again ‘protect’ the source from non-local influences, by making sure that the settings of the polarizers be in the absolute future of the emission at the source. Admittedly, it would be extremely difficult to perform experiments of this kind, particularly in those cases in which the emission at the source constitutes an stochastic process. Nevertheless, as a matter of principle, it is not quite unreasonable to assume that the emission at the source be uninfluenced by the settings of distant polarizers.

Finally, we should stress that the ‘free will assumption’ is not an issue here, for similar reasons as those just discussed. According to the free will assumption, an operator is free to choose between different measurement settings [4], i.e., choices are not predetermined by some initial conditions. We will consider experimental situations like the one addressed by Gröblacher *et al* in which the settings of the instruments are not changed during the flight of the particles. If these settings are fixed ‘sufficiently in advance to allow them to reach some mutual rapport by exchange of signals with velocity less than or equal to that of light’ [5], contextual models—as already said—may well be in accordance with relativity. If, in contrast, the settings were changed by an operator during the flight of the particles, then contextual models could violate relativistic causality. A way to avoid this possible conflict with relativity is to relax the free will assumption and consider that the actual settings of the instruments by an operator have been fixed in the past by some initial conditions, thereby making free will a mere illusion. The models considered here do not address the question of free will, because they assume fixed settings.

The paper is organized as follows. Section 2 begins with a discussion of the experiment of Gröblacher *et al*. In subsection 2.1 we introduce the Kochen–Specker model for a single qubit in a form which is amenable to generalization. Building upon this model we present in subsection 2.2 an extension that works for two qubits. This is the central result of the paper. Finally, in section 3 we present our conclusions.

2. The Kochen–Specker model and its extension to the non-local case

Gröblacher *et al* [1] carried out experiments with two entangled qubits which were realized as polarization entangled photons generated via spontaneous parametric down-conversion.

The contextual models that were put to the test should satisfy the following assumptions: (1) measurement outcomes reveal pre-existing properties (realism[2]); (2) physical states are mixtures of subensembles with definite properties (e.g., polarization); (3) the expectation values taken for each subensemble coincide with the predictions of QM (e.g., polarization states obey Malus' law).

Consider a source emitting pairs of photons toward two measuring devices whose respective settings, as given by unit vectors n_a and n_b, are fixed by Alice and Bob in each run of the experiment. The emitted photons have well-defined polarizations n_u and n_v. Let us denote by A and B the outcomes for polarization measurements along n_a and n_b, respectively. Their values are ± 1, corresponding to transmission/absorption of a photon. By writing, e.g., $A = A(\lambda, n_a, n_b) \equiv A(\lambda, a, b)$ we make explicit reference to the assumption that Alice's results may depend on Bob's settings and that the model is, thus, contextual. Let $\rho_{uv}(\lambda)$ be the subensemble distribution of the photon pairs emitted by the source with polarizations n_u and n_v. It is required that, according to Malus' law, local averages satisfy

$$\begin{aligned} \overline{A} &= \int \mathrm{d}\lambda\, \rho_{uv}(\lambda) A(\lambda, a, b) = n_u \cdot n_a, \\ \overline{B} &= \int \mathrm{d}\lambda\, \rho_{uv}(\lambda) B(\lambda, b, a) = n_v \cdot n_b. \end{aligned} \tag{1}$$

For a source emitting well-polarized photon pairs the correlation function of measurement results is given by $\overline{AB}(u, v) = \int \mathrm{d}\lambda\, \rho_{u,v}(\lambda) A(\lambda, a, b) B(\lambda, b, a)$. For a more general source, we assume to have at our disposal a distribution function $F(u, v)$ that describes the mixture of polarized pairs produced by the source. The general correlation function is thus given by $E_{ab} = \langle AB \rangle = \int \mathrm{d}u\, \mathrm{d}v\, F(u, v) \overline{AB}(u, v)$. So far, we have followed the assumptions and, closely, the notation of Gröblacher *et al*. These authors derived a Leggett-type inequality to be experimentally tested. The said inequality follows from the identities $-1 + |A + B| = AB = 1 - |A - B|$, which are fulfilled whenever $A = \pm 1$ and $B = \pm 1$. Multiplying these identities by $\rho_{uv}(\lambda)$, integrating over λ and noting that $\overline{|A \pm B|} \geqslant |\overline{A} \pm \overline{B}|$, one obtains $-1 + |\overline{A} + \overline{B}| \leqslant \overline{AB} \leqslant 1 - |\overline{A} - \overline{B}|$. Further multiplication by$F(u, v)$ and subsequent integration over (u, v) leads to a Leggett-type inequality that puts a bound on the correlations E_{ab} predicted by a contextual HV model. QM predicts $E_{ab} = -n_a \cdot n_b$ and these values violate the said inequality. The experiments of Gröblacher *et al* reproduce the quantum-mechanical predictions with high accuracy. Now, as we observe from the above definitions, contextuality has been restricted to the measurement outcomes of Alice and Bob, as if the source could not be affected by, e.g., Bob's settings, while A could be affected by such settings. This is, of course, an arbitrary assumption from the viewpoint of a contextual model. All the more, if we recall that what is considered a 'source' (or part of it) in one experiment, may be dubbed as 'measuring device' in the other. A contextual model should consistently assume functions of the form $\rho_{uv}(\lambda, a, b)$ and $F(u, v, a, b)$. Though the derivation of the Legget-type inequality presented in [1] appears to be insensitive to a possible contextuality of ρ_{uv}, a distribution function of the form $F(u, v, a, b)$ invalidates the derivation of the Legget-type inequality of [1] (see supplementary information to [1, 3]). Hence, the contextual models that have been tested in the experiments of Gröblacher *et al* belong to a restricted class. This class would be defined through the nonphysical assumption that sources and measuring devices are differently influenced by the experimental context, the sources being somehow 'protected' from contextual influences. In fact, it is possible to construct an explicit contextual model that reproduces the experimental results reported in [1], as we shall see next. The construction is

[2] I use the term 'realism' as it is defined by Gröblacher *et al*. Alternatively, for some authors 'realism' can be defined so as to allow that measurement results be generated at the moment of measurement.

based on the Kochen–Specker (KS) model for a single qubit [6, 7]. We will first discuss the KS model, but in a way that departs from its original formulation. Our approach has been tailored in a way that can be generalized to the two-qubit case.

2.1. The Kochen–Specker model for a single qubit

In the KS model, the HVs λ span the unit sphere S^2. Thus, λ can be parameterized as a unit vector $n_\lambda = (\sin\theta_\lambda \cos\varphi_\lambda, \sin\theta_\lambda \sin\varphi_\lambda, \cos\theta_\lambda)$. Henceforth we will write, interchangeably, λ or n_λ, for these and all other unit vectors playing the role of HVs. Now, S^2 is also the Bloch sphere, which serves to represent geometrically a qubit $|\psi\rangle = \cos(\theta_\psi/2)\,\mathrm{e}^{-\mathrm{i}\varphi_\psi/2}|+\rangle + \sin(\theta_\psi/2)\,\mathrm{e}^{\mathrm{i}\varphi_\psi/2}|-\rangle$, where $|\pm\rangle$ are the S_z-eigenvectors. Indeed, we can assign to each $|\psi\rangle$ a unit vector $n_\psi = (\sin\theta_\psi \cos\varphi_\psi, \sin\theta_\psi \sin\varphi_\psi, \cos\theta_\psi)$ on the Bloch sphere and, reciprocally, to each unit vector $n_\psi \in S^2$ it corresponds a normalized qubit state $|\psi\rangle$ (modulo a phase). The KS model assigns to each qubit $|\psi\rangle$ a probability density $\rho_\psi(\lambda)\,\mathrm{d}\lambda$, with $\mathrm{d}\lambda$ being a suitable measure. This corresponds to a completion through HVs of the supposedly 'incomplete' description of physical reality that is provided by the state vector $|\psi\rangle$. Consider now Alice's observable $\widehat{A}$. As is well known, it can be written in the form $\widehat{A} = a_0 + \mathbf{a}\cdot\sigma$, where σ stands for the triple of Pauli matrices. The eigenvectors of $\widehat{A}$ will be denoted by $\left|\psi_a^\pm\right\rangle$. They are the same as those of $n_a\cdot\sigma$, with $n_a = \mathbf{a}/|\mathbf{a}|$. To projection operators like $\widehat{\Pi}_a^\pm = \left|\psi_a^\pm\right\rangle\!\left\langle\psi_a^\pm\right|$ we assign characteristic dichotomic functions, $\chi_a^\pm(\lambda)$, each taking the values 1 and 0, in correspondence to whether a given event does or does not take place. For example, $\chi_a^+(\lambda)$ states whether a measurement of $\widehat{A}$ along n_a does or does not produce a 'positive' result (e.g., detection along the upward direction in a Stern–Gerlach set-up), and similarly for $\chi_a^-(\lambda)$ (counting now as 'positive' a detection along the downward direction in the Stern–Gerlach set-up). That is, if $\chi_a^+(\lambda) = 1$, then $\chi_a^-(\lambda) = 0$, and vice versa. Hence, we can write $A(\lambda) = \chi_a^+(\lambda) - \chi_a^-(\lambda)$. The functions $\rho_\psi(\lambda)\,\mathrm{d}\lambda$ and $\chi_a^\pm(\lambda)$ should be chosen so as to afford, for all $|\psi\rangle$, that

$$\langle\psi|\widehat{\Pi}_a^\pm|\psi\rangle = \int \rho_\psi(\lambda)\chi_a^\pm(\lambda)\,\mathrm{d}\lambda. \tag{2}$$

The model then reproduces all quantum-mechanical predictions about probabilities of measurement outcomes, in the sense that for all qubits $|\psi\rangle$ and for any operator $\widehat{A}$, it holds true that $\langle\psi|\widehat{A}|\psi\rangle = \int \rho_\psi(\lambda)A(\lambda)\,\mathrm{d}\lambda$. This suffices for our present purposes, although, if required, we could also prescribe the dynamics between measurements. Equation (2) will be satisfied if for any $|\psi_a\rangle$ and $|\psi_b\rangle$, such that $n_i\cdot\sigma|\psi_i\rangle = |\psi_i\rangle$, $i = a, b$, we define $\rho_a(\lambda)\,\mathrm{d}\lambda$ and $\chi_b(\lambda)$ so that $|\langle\psi_b \mid \psi_a\rangle|^2 = \langle\psi_a|\widehat{\Pi}_b|\psi_a\rangle = \langle\psi_b|\widehat{\Pi}_a|\psi_b\rangle = \cos^2(\theta_{ab}/2) = \int \rho_a(\lambda)\chi_b(\lambda)\,\mathrm{d}\lambda$, where $\theta_{ab} = \cos^{-1}(n_a\cdot n_b)$. The following definition, due to KS, satisfies our requirements and serves as a basis for handling the two-qubit case. It is based on the fact that each unit vector divides S^2 in two hemispheres. We define $\rho_a(\lambda)$ as being different from zero only on the intersection of the northern hemispheres of n_a and $n_\lambda = \lambda$, where it takes the value $n_\lambda\cdot n_a/\pi$. This can be expressed with the help of Heaviside's step function ($\Theta(x) = 1$, for $x \geqslant 0$ and $\Theta(x) = 0$, for $x < 0$) as

$$\rho_a(\lambda) = \frac{n_\lambda\cdot n_a}{\pi}\Theta(n_\lambda\cdot n_a). \tag{3}$$

On the other hand, the characteristic functions are defined as

$$\chi_i^\pm(\lambda) = \Theta\big(n_\lambda\cdot n_i^\pm\big), \qquad i = a, b. \tag{4}$$

Here, $n_b^\pm$—similarly to $n_a^\pm$—are in one-to-one correspondence with the eigenvectors $\left|\psi_b^\pm\right\rangle$ of Bob's observable $\widehat{B} = b_0 + \mathbf{b}\cdot\sigma$. Next, we show that $\int \rho_a(\lambda)\chi_b^+(\lambda)\,\mathrm{d}\lambda = \cos^2(\theta_{ab}/2)$

and $\int \rho_a(\lambda)\chi_b^-(\lambda)\,\mathrm{d}\lambda = \sin^2(\theta_{ab}/2)$, as desired. The measure is taken to be $\mathrm{d}\lambda \equiv \mathrm{d}S_\lambda = \sin\theta_\lambda\,\mathrm{d}\theta_\lambda\,\mathrm{d}\varphi_\lambda$, the surface element on the unit sphere.

Let us take $n_a = n_a^+ \leftrightarrow |\psi_a^+\rangle$ and $n_b = n_b^+ \leftrightarrow |\psi_b^+\rangle$, for concreteness. Note that, because $\langle\psi_a|\widehat{\Pi}_b|\psi_a\rangle = \langle\psi_b|\widehat{\Pi}_a|\psi_b\rangle$, it should hold true that $I_{ab} \equiv \int \rho_a(\lambda)\chi_b(\lambda)\,\mathrm{d}\lambda = \int \rho_b(\lambda)\chi_a(\lambda)\,\mathrm{d}\lambda \equiv I_{ba}$. Writing I_{ab} more explicitly we obtain

$$I_{ab} = \frac{1}{\pi}\int n_\lambda \cdot n_a \Theta(n_\lambda \cdot n_a)\Theta(n_\lambda \cdot n_b)\,\mathrm{d}\lambda = \frac{1}{\pi}\int_{N_a\cap N_b} n_a \cdot n_\lambda\,\mathrm{d}S_\lambda, \tag{5}$$

where $N_a \cap N_b \equiv S_{ab}$ is the area over which the integration is effectively restricted by $\Theta(n_\lambda \cdot n_a)\Theta(n_\lambda \cdot n_b)$. N_a and N_b are the northern hemispheres belonging to the Poles n_a and n_b, respectively. Now, the last expression in equation (5) expresses I_{ab} as a flux integral. Defining the vector field $v_a(r) = n_a \times r/2$, we have $n_a = \nabla \times v_a(r)$, so that, by applying Stoke's theorem, we obtain

$$\begin{aligned} I_{ab} &= \frac{1}{\pi}\int_{S_{ab}} \nabla \times v_a(r) \cdot n_\lambda\,\mathrm{d}S_\lambda = \frac{1}{\pi}\oint_{\partial S_{ab}} v_a(r_\lambda)\cdot \mathrm{d}r_\lambda \\ &= \frac{1}{2\pi}\oint_{\partial S_{ab}} n_a \times r_\lambda \cdot \mathrm{d}r_\lambda = \frac{1}{2\pi}\oint_{\partial S_{ab}} \left(r_\lambda \times \frac{\mathrm{d}r_\lambda}{\mathrm{d}s}\cdot n_a\right)\mathrm{d}s, \end{aligned} \tag{6}$$

where ∂S_{ab} means the contour limiting S_{ab} and $\mathrm{d}s$ is the arc length used to parameterize the curve $r_\lambda(s) \equiv n_\lambda(s)$ on S^2, so that $\mathrm{d}r_\lambda/\mathrm{d}s$ is a unit-vector tangent to the sphere. The contour ∂S_{ab} limiting S_{ab} is made of two great circles, C_a and C_b, the 'equators' relative to n_a and n_b, respectively. They intersect at two antipodal points, P_1 and P_2, say. The contour integral can thus be split into two line integrals, one going from P_1 to P_2 along C_a, and the other from P_2 back to P_1 along C_b. Each of these curves is half a great circle and has thus a length equal to π. Now, $r_\lambda \times \mathrm{d}r_\lambda/\mathrm{d}s$ is also a unit vector—the so-called bivector in the theory of curves—which equals n_a along C_a and n_b along C_b. Whence,

$$\begin{aligned} I_{ab} &= \frac{1}{2\pi}\left(\int_{C_a} n_a \cdot n_a\,\mathrm{d}s + \int_{C_b} n_b \cdot n_a\,\mathrm{d}s\right) \\ &= \frac{1}{2}(1 + n_b \cdot n_a) = \cos^2\left(\frac{\theta_{ab}}{2}\right). \end{aligned} \tag{7}$$

The case $n_b = n_b^- = -n_b^+$ gives $I_{ab} = (1 - n_b \cdot n_a)/2 = \sin^2(\theta_{ab}/2)$, and this completes the proof. Our procedure also makes clear how the symmetry under $n_a \leftrightarrow n_b$ arises, so that $I_{ab} = I_{ba}$, as already mentioned.

2.2. A Kochen–Specker model for the two-qubit case

Let us now turn to the two-qubit case, specifically addressing the experiment reported by Gröblacher *et al* [1]. Our aim is to give a counterexample for a 'no-go' assertion which states that no contextual HV model—within a wide class—would be capable of explaining the results of the said experiment. If this assertion were true, then a wide class of contextual HV models would have been ruled out by the experiment. As already mentioned, this experiment was conceived as a test of a Leggett-type inequality that was derived from the following assumptions: (i) in experiments using a source that emits pairs of photons with well-defined polarizations n_u and n_v, each emitted pair belongs to a subensemble that is defined through a density ρ_{uv}. Measurements performed by Alice and Bob—possibly influencing each other in a non-local way—produce outputs obeying Malus' law, equation (1). (ii) For a general source producing mixtures of polarized photons, there is a function $F(u, v)$, ruling the distribution of polarizations. Under these circumstances, a Leggett-type inequality should hold true [1]. However, QM violates this inequality and is in accordance with the results obtained in the experiment [1]. The source that was used in the experiment produced polarization-entangled singlet states $|\Psi^-\rangle_{AB} = (|H\rangle_A|V\rangle_B - |V\rangle_A|H\rangle_{AB})/\sqrt{2}$ of vertically (V) and horizontally (H) polarized photons, by means of a standard type-II parametric down-conversion process. We construct next a contextual model satisfying the above requirements and being in accordance with the predictions of QM. This model will be tailored so as to reproduce the results of the particular experiment we have in sight, i.e., we will assume an initial distribution that corresponds to the singlet state $|\Psi^-\rangle_{AB}$. To this end, let us first set $\rho_{uv}(\lambda_1,\lambda_2) = \rho_u^+(\lambda_1)\rho_v^+(\lambda_2)$, where ρ_u^+, ρ_v^+ are defined as in equation (3). We will see that this choice enforces Malus' law. Thereafter, we will choose an appropriate contextual distribution $F(u,v)$ that describes the initial state. Note that we have divided the HVs into two groups: $\lambda = (\lambda_1,\lambda_2)$. A possible interpretation of this division is that λ_1 relates to Alice's measuring device and λ_2 to Bob's. This interpretation would be consistent with the assumption of contextual influences acting upon the source. Alternatively, one could take $\lambda_{1,2}$ to be parameters that are carried by the particles that are registered by Alice and Bob. In such a case, ρ_{uv} would have been locally defined. It is straightforward to see that our ρ_{uv} satisfies Malus' law. Indeed, $\int \mathrm{d}\lambda_2\,\mathrm{d}\lambda_1 \rho_{uv}\chi_a^{\pm}(\lambda_1) = \int \mathrm{d}\lambda_2\,\rho_v^+(\lambda_2)\int \mathrm{d}\lambda_1\,\rho_u^+(\lambda_1)\chi_a^{\pm}(\lambda_1) = (1 \pm n_u.n_a)/2$, so that $\overline{A} = \int \mathrm{d}\lambda\,\rho_{uv}A(\lambda_1) = \int \mathrm{d}\lambda\,\rho_{uv}\big(\chi_a^+(\lambda_1) - \chi_a^-(\lambda_1)\big) = n_u \cdot n_a$. Here, we have used the results of the first part, i.e., that $\rho_u^+(\lambda)$ is normalized, and equation (7) with $n_b \to n_u$. Similarly, one obtains $\overline{B} = \int \mathrm{d}\lambda\,\rho_{uv}B(\lambda_2) = n_v \cdot n_b$. Because our ρ_{uv} factorizes, then $\overline{AB}(u,v) = \overline{A}\cdot\overline{B} = (n_u \cdot n_a)(n_v \cdot n_b)$, so that

$$E_{ab} = \langle AB\rangle = \int \mathrm{d}u\,\mathrm{d}v\,F(u,v)\overline{AB}(u,v) = \int \mathrm{d}u\,\mathrm{d}v\,F(u,v)(n_u\cdot n_a)(n_v\cdot n_b). \tag{8}$$

Let us now take a contextual distribution function which is appropriate for our scopes. A possible choice is the following one:

$$F_{ab}(u,v) = \frac{1}{2\pi^2}\big(\chi_a^+(\lambda_u)\chi_a^-(\lambda_v) + \chi_b^-(\lambda_u)\chi_b^+(\lambda_v)\big). \tag{9}$$

With this $F_{ab}(u,v)$ we obtain the desired result, i.e., that $E_{ab} = \langle AB\rangle = -n_a\cdot n_b$, in accordance with the quantum-mechanical prediction. Indeed, replacing equation (9) in equation (8) E_{ab} can be written as the sum of two terms: $E_{ab} = (I_a + I_b)/2$, with $I_j = \int \mathrm{d}\lambda_u\,\mathrm{d}\lambda_v(n_u\cdot n_a)(n_v\cdot n_b)\chi_j^+(\lambda_u)\chi_j^-(\lambda_v)/\pi^2, (j = a, b)$. We can calculate the two integrals following a similar procedure as we did before:

$$\begin{aligned} I_a &= \frac{1}{\pi^2}\int \mathrm{d}\lambda_u\,\mathrm{d}\lambda_v(n_u\cdot n_a)(n_v\cdot n_b)\chi_a^+(\lambda_u)\chi_a^-(\lambda_v) \\ &= \overbrace{\frac{1}{\pi}\int \mathrm{d}\lambda_u\,\chi_a^+(\lambda_u)(n_u\cdot n_a)}^{1}\int \mathrm{d}\lambda_v(n_v\cdot n_b)\frac{\chi_a^-(\lambda_v)}{\pi} \\ &= \frac{1}{\pi}\int_{S_a}\nabla\times v_b(r_\lambda)\cdot n_\lambda\,\mathrm{d}S_\lambda \\ &= \frac{1}{2\pi}\oint_{C_a} n_b\times r_\lambda\cdot\mathrm{d}r_\lambda \\ &= \frac{1}{2\pi}\oint_{C_a} r_\lambda\times\frac{\mathrm{d}r_\lambda}{\mathrm{d}s}\cdot n_b\,\mathrm{d}s \\ &= -n_a\cdot n_b. \end{aligned} \tag{10}$$

Here, S_a is the southern hemisphere of n_a, C_a its equator, and $v_b(r) = n_b\times r/2$. We have applied Stokes' theorem as in equation (6), but now the contour C_a is oriented clockwise. In a similar way one obtains $I_b = \int \mathrm{d}\lambda_u\, \mathrm{d}\lambda_v (n_u \cdot n_a)(n_v \cdot n_b)\chi_b^+(\lambda_u)\chi_b^-(\lambda_v)/\pi^2 = -n_a \cdot n_b$, so that $E_{ab} = \langle AB\rangle = (I_a + I_b)/2 = -n_a \cdot n_b$, as desired. Note that by taking $F_a(u, v) = \left(\chi_a^+(\lambda_u)\chi_a^-(\lambda_v)\right)/\pi^2$ we would have obtained the same result. We chose F_{ab} for the sake of symmetry between the two parties. In any case, we have succeeded in constructing a counterexample, i.e., a contextual HV model that reproduces the predictions of QM for a particular case, namely, the experiment performed by Gröblacher *et al*.

Although we have worked out in detail a particular case, it is clear that our procedure could be extended so as to mimic QM in a general case. That is, for cases in which the initial state is not the singlet state, we could also construct a contextual HV model. The singlet state is one of the four states that constitute the standard Bell basis, which is given by $|\Psi^\pm\rangle = (|H\rangle_A|V\rangle_B \pm |V\rangle_A|H\rangle_{AB})/\sqrt{2}$ and $|\Phi^\pm\rangle = (|V\rangle_A|V\rangle_B \pm |H\rangle_A|H\rangle_{AB})/\sqrt{2}$. We have seen that $\langle\Psi^-|AB|\Psi^-\rangle = -n_a \cdot n_b$. For the other Bell states, $\langle AB\rangle$ can be expressed in terms of the components of $n_a = (a_x, a_y, a_z)$ and $n_b = (b_x, b_y, b_z)$ as $\langle\Psi^+|AB|\Psi^+\rangle = a_xb_x + a_yb_y - a_zb_z$, $\langle\Phi^+|AB|\Phi^+\rangle = a_xb_x - a_yb_y + a_zb_z$ and $\langle\Phi^-|AB|\Phi^-\rangle = -a_xb_x + a_yb_y + a_zb_z$. Now, it is clear from our above results that we can easily choose an appropriate distribution $F_{ab}(u, v)$ for all Bell states, as we did for $|\Psi^-\rangle$. Indeed, take for example the state $|\Psi^+\rangle$. We can write $\langle\Psi^+|AB|\Psi^+\rangle = (n_a \cdot e_x)(n_b \cdot e_x) + (n_a \cdot e_y)(n_b \cdot e_y) - (n_a \cdot e_z)(n_b \cdot e_z)$, with e_x, e_y, e_z being the unit vectors with respect to which we have defined the coordinates of n_a and n_b. Having expressed $\langle\Psi^+|AB|\Psi^+\rangle$ in terms of scalar products, it is straightforward to choose $F_{ab}(u, v)$ by looking at the derivation of equation (10). Indeed, from an analogous calculation we can readily prove that $\int \mathrm{d}\lambda_u\, \mathrm{d}\lambda_v (n_u \cdot n_a)(n_v \cdot n_b)\chi_i^+(\lambda_u)\chi_j^\pm(\lambda_v)/\pi^2 = \pm(n_a \cdot e_i)(n_b \cdot e_j)$, $i, j = x, y, z$. Hence, we can choose $F_{ab}(u, v)$ so as to obtain any desired combination of scalar products when we insert it into equation (8). A general, initial state $|\Psi\rangle$ can be written as a linear combination of the Bell states. One can then easily check that $\langle\Psi|AB|\Psi\rangle$ contains only binary products of the Cartesian components of n_a and n_b. Thus, the above result applies for the general case of an arbitrary initial state $|\Psi\rangle$. The case of an initial mixed state can be dealt with by writing a combination of different distributions $F_{ab}(u, v)$ with appropriate weights.

3. Conclusions

In view of the two-qubit model we have discussed, we can draw the following conclusions. Our model reproduces the predictions of QM for the experiment of Gröblacher *et al* and hence violates the Legget-type inequality derived in [1]. Though one can qualitatively consider other, simpler realistic non-local models that are not addressed by the Leggett inequality [9], the one presented here is an explicit one, that is very akin to those considered by Leggett and by Gröblacher *et al*. The point of departure from the derivation presented in [1] is that we considered a contextual distribution $F_{ab}(u, v)$ instead of the non-contextual $F(u, v)$ that was assumed in [1, 3]. Within a contextual theory it is certainly justified to take $F_{ab}(u, v)$ together with contextual densities $\rho_{uv}(\lambda, a, b)$. Moreover, as already mentioned, contextual densities or distributions would not be extraneous to a classical approach. They could be thought of as arising from solving some fundamental partial differential equations. To prescribe 'boundary conditions' in order to solve these equations would be tantamount to allow 'contextuality'. We can therefore conclude that the experiments recently reported by Gröblacher *et al* [1] do not address a broad plausible class of contextual, hidden-variable models. They did address a class whose defining feature requires that models pertaining to it may ascribe contextual qualities to the measuring devices, but not to those devices that were included as part of a 'source'. Being deliberately provocative, let us illustrate the said defining feature by referring to quantum optics experiments. Advocates of the aforementioned class must contend that even though the action of a calcite crystal (in front of a detector) could be affected by distant devices, the action of a beta-barium-borate crystal (in a source) could not. This can hardly be an assumption worth to be tested. Instead of conducting experiments of increased refinement [8] to test realistic models of a class that could have been discarded from the outset, it would be more meaningful to introduce, for instance, variable polarizers as a tool for testing non-locality. Consider for instance an experiment with variable polarizers that yields the results predicted by QM. The model presented here could be slightly modified so as to explain such results, but at the cost of entering into the class of highly counterintuitive ones. Let us finally mention that by properly testing contextuality in physically plausible HV models one could complement some recently proposed experiments that address determinism together with non-contextuality [10, 11]. If these experiments do confirm the predictions of QM, then contextuality should be necessarily included in any fundamental description of physical phenomena, unless we are ready to admit an inherent indeterminacy in the ultimate nature of these phenomena. In other words, we would be faced with the choice between contextuality and indeterminism.

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